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Compact Finite Difference Schemes for Solving a Class of Weakly-Singular Partial Integro-differential Equations

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Abstract:: : In this paper, we present a new approach to resolve linear weakly-singular partial integro-differential equations by first removing the singularity using Taylor's approximation and transform the given partial integro-differential equations into an partial differential equation. After that the fourth order compact finite difference scheme and collocation method is presented to obtain system of algebraic equations which solved to compute the unknown function. The efficiency and accuracy of the method is validated by its application to several distinct test problems which have exact solutions..

Keywords: T weakly-singular partial integro-differential equations; Taylor's approximation; fourth order compact finite difference scheme; collocation method.

Any functional equation in which the unknown function appears under the sign of integration is called an integral equation. In many instances the integral equation originates from the conversion of a boundary-value problem or an initial-value problem associated with a partial or an ordinary differential equation. Integral equations arise in a great many branches of science; for example, in potential theory, acoustics, elasticity, fluid mechanics, raditive transfer theory of population, etc. Also integro-differential equations (IDEs) arise widely in mathematical models of certain biological and physical phenomena.
In this paper we study the linear weakly-singular parabolic integro-differential equations with a memory term:

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\int_0^t k(t-s)\,u(x,s)\,ds+f(x,t), \qquad x\in(0,1), \quad t\in[0,T], \tag{1.1}$$

It is associated by the boundary and initial conditions

$$u(0,t)=g_1(t), \quad u(1,t)=g_2(t), \quad t\geq 0, \tag{1.2}$$

$$u(x,0)=u_0(x), \qquad 0\leq x\leq 1, \tag{1.3}$$

where $k(t)=t^{-\alpha}$.

The solution $u(.,t)$ and the source term $f(.,t)$ take values in $L_2([0,1])$, and the initial data u_0 is an element of $L_2([0,1])$. Equations of type (1.1) may be thought of as a model problem occurring in the theory of heat conduction in materials with memory, population dynamics and viscoelasticity (e.g., Friedman & Shinbrot, 1967 [5]; Heard, 1982 [6]; Renardy et al., 1987 [11]). Due to the wide application of these equations, they must be solved successfully with efficient numerical methods. Many authors have considered numerical methods for a linear problem of the form (1.1). Typically, the time discretization is affected by a combination of finite difference and quadratures. Finite difference in time and finite elements in space have been discussed in the case of a smooth kernel (e.g., Sloan & Thomée, 1986 [12]; Cannon & Lin, 1988, 1990 [2,3]; Yanik & Fairweather, 1988 [15]; Thomée & Zhang, 1989 [14]; Lin et al., 1991 [9]; Zhang, 1993 [16]; Mclean, Thomée. & Wahlbin (1996) [10]). For the nonsmooth kernel case we refer to Chen et al. (1992) [4] and Larsson et al. (1998) [8]. Collocation method has been discussed in (Te Riele (1982) [13]; Brunner, Pedas, & Vainikko, (2001) [1])
Our contribution in this paper is to develop a new approach to resolve linear weakly-singular partial integro-differential equations in one dimensional space with non-homogeneous Dirichlet boundary conditions. The suggested numerical scheme starts with removing the singularity using Taylor's approximation and transforming the given second-order partial integro-differential equations into partial differential equation, then we use the discretization in

time by the 2-point Euler backward finite difference method. After that we use compact finite difference scheme and then we use a collocation method to obtain system of algebraic equations which solved to compute the unknown function. The proposed techniques are programmed using Matlab ver. 7.8.0.347 (R2009a).

The paper is organized as follows: In Section 2, we introduce a method of solution for weakly-singular partial integro-differential equations with varying boundary conditions by using Taylor's approximation, then we give a brief introduction to a high accurate compact finite difference formula for partial differential equations with varying boundary conditions, then we use collocation method. In Section 3, the proposed method is directly applicable to solve several distinct numerical examples to support the efficiency of the suggested numerical method. Conclusions are drawn in Section 4.

1. Method of Solution

We propose an approximate solution for solving weakly-singular partial integro-differential equations. The advantage of this method is that we remove the singularity of the kernel of weakly-singular partial integro-differential equations at $s=t$ by judiciously applying Taylor's approximation and then transforming the given weakly-singular partial integro-differential equation into partial differential equation. Next, the discretization in time by the 2-point Euler backward differentiation formula is manipulated to convert the partial differential equation into ordinary differential equation. Then we use compact finite difference scheme and collocation method to obtain system of algebraic equations which solved to compute the unknown function.

2.1 Taylor's approximation

Consider the following weakly-singular integro-differential equation partial

$$u_t - u_{xx} = \int_0^t \frac{u(x,s)}{(t-s)^{\alpha}} ds + f(x,t). \tag{2.1.1}$$

It is associated by the boundary and initial conditions

$$u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \quad t \ge 0, \tag{2.1.2}$$

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1. \tag{2.1.3}$$

Rewrite equation (2.1.1) as:

$$u_t - u_{xx} = \int_0^t \frac{u(x,s) - u(x,t) + u(x,t)}{(t-s)^{\alpha}} ds + f(x,t) \tag{2.1.4}$$

$$u_t - u_{xx} = \int_0^t \frac{u(x,t)}{(t-s)^{\alpha}} ds + \int_0^t \frac{u(x,s) - u(x,t)}{(t-s)^{\alpha}} ds + f(x,t), \tag{2.1.5}$$

equivalently

$$u_t - u_{xx} = u(x,t)\frac{t^{(1-\alpha)}}{(1-\alpha)} - \int_0^t (t-s)^{1-\alpha} \frac{u(x,s) - u(x,t)}{(s-t)} ds + f(x,t). \tag{2.1.6}$$

By using Taylor's approximation of $u(x,s)$ about $s=t$,

$$u(x,s) \approx u(x,t) + (s-t)\, u_t(x,t), \tag{2.1.7}$$

or

$$\frac{u(x,s) - u(x,t)}{(s-t)} \approx u_t(x,t). \tag{2.1.8}$$

From equation (2.1.8) into equation (2.1.6), then

$$u_t(x,t) - u_{xx}(x,t) = u(x,t)\frac{t^{(1-\alpha)}}{(1-\alpha)} - u_t(x,t)\int_0^t (t-s)^{1-\alpha} ds + f(x,t), \tag{2.1.9}$$

so

$$u_t(x,t)-u_{xx}(x,t)=u(x,t)\frac{t^{(1-\alpha)}}{(1-\alpha)}-u_t(x,t)\frac{t^{(2-\alpha)}}{(2-\alpha)}+f(x,t), \tag{2.1.10}$$

equivalently

$$\left[1+\frac{t^{(2-\alpha)}}{(2-\alpha)}\right]u_t(x,t)-u_{xx}(x,t)-\frac{t^{(1-\alpha)}}{(1-\alpha)}u(x,t)=f(x,t). \tag{2.1.11}$$

2.3 Compact Finite Difference Schemes

In this section, we give a brief introduction to a high accurate compact finite difference formula for partial differential equations with varying boundary conditions.

2.3.1 Formulation of High-Order Compact Schemes

Compact Schemes are based on a fourth order accurate approximation to the derivative calculated from ordinary differential equation. To developed the scheme for one-dimensional uniform Cartesian grids with spacing $\Delta x=h$, let us introduce the following notations [7]: If $u_j \equiv u(x_j)$, then we use notations

$$\delta_+ u_j=\frac{u_{j+1}-u_j}{h}=\delta_{x_+}, \quad \delta_- u_j=\frac{u_j-u_{j-1}}{h}=\delta_{x_-}, \tag{2.3.1.1}$$

to denote the standard forward finite difference and backward finite difference schemes for first derivative. Also,

$$\delta_0 u_j=\tfrac{1}{2}\left(\delta_+ u_j+\delta_- u_j\right)=\frac{u_{j+1}-u_{j-1}}{2h}, \tag{2.3.1.2}$$

is the first-order central finite difference with respect to *x*. The standard second-order central finite difference is denoted as $\delta_x^2 u_j$ and is defined as

$$\delta_+ \delta_- u_j=\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}=\frac{\delta_+-\delta_-}{h}. \tag{2.3.1.3}$$

By using the Taylor's series expansion, a fourth and sixth orders accurate finite difference for the first and second derivatives can be approximated by

$$\delta_0 u=\frac{du}{dx}+\frac{h^2}{3!}\frac{d^3u}{dx^3}=\left(1+\frac{h^2}{6}\frac{d^2}{dx^2}\right)\frac{du}{dx}=\left(1+\frac{h^2}{6}\delta^2\right)\frac{du}{dx}+O(h^4) \tag{2.3.1.4}$$

and

$$\delta_x^2 u=\frac{d^2u}{dx^2}+\frac{h^2}{12}\frac{d^4u}{dx^4}=\left(1+\frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{d^2u}{dx^2}=\left(1+\frac{h^2}{12}\delta^2\right)\delta^2u+O(h^4), \tag{2.3.1.5}$$

2.3.2 Compact finite difference method for solving partial differential equations

Here, we use the fourth order compact finite difference method to solve problem (2.1.11) with boundary and initial conditions (2.1.2, 2.1.3). First the discretization in time by the 2-point Euler backward differentiation formula is manipulated to convert the partial differential equation into ordinary differential equation. To construct a numerical solution, we first consider the nodal points (x_j,t_i) defined in the region $[0,1]\times[0,T]$ where

$$0=x_0<x_1<\cdots<x_{n-1}<x_n=1, \qquad x_{j+1}-x_j=h,$$

and

$$0=t_0<t_1<\cdots<t_i<\cdots<T, \qquad t_{i+1}-t_i=\tau.$$

In such a case we have $x_j=jh$ for $j=0,1,2,\ldots,n,$ and $t_i=i\tau$ for $i=0,1,2,\ldots$.

Next, the 2-point Euler backward differentiation formula is manipulated to approximate u_t, given in equation (2.1.11), at the time-level t_{i+1} for $i=0,1,2,\ldots$. Therefore, we have

$$\left[1+\frac{t^{(2-\alpha)}}{(2-\alpha)}\right]\left[\frac{u_{i+1}(x)-u_i(x)}{\tau}\right]-\frac{d^2u_{i+1}(x)}{dx^2}-\frac{t^{(1-\alpha)}}{(1-\alpha)}u_{i+1}(x)=f_{i+1}(x)\,, \tag{2.3.2.1}$$

equivalently

$$\left[1+\frac{t^{(2-\alpha)}}{(2-\alpha)}-\frac{\tau\, t^{(1-\alpha)}}{(1-\alpha)}\right]u_{i+1}(x)-\tau\, u''_{i+1}(x)=\tau\, f_{i+1}(x)+\left[1+\frac{t^{(2-\alpha)}}{(2-\alpha)}\right]u_i(x) \tag{2.3.2.2}$$

where $f_{i+1}(x)=f(x,t_{i+1})$, $u_i(x)=u(x,t_i)$ and $u_{i+1}(x)=u(x,t_{i+1})$.

Putting $x=x_j,\quad j=1,\cdots,n-1$ in (2.3.2.2), then

$$u''_{i+1,\,j}=\left[\frac{1}{\tau}+\frac{t^{(2-\alpha)}}{\tau(2-\alpha)}-\frac{t^{(1-\alpha)}}{(1-\alpha)}\right]u_{i+1,\,j}-f_{i+1,\,j}-\left[\frac{1}{\tau}+\frac{t^{(2-\alpha)}}{\tau(2-\alpha)}\right]u_{i,\,j}\,,$$

(2.3.2.3)

let

$$a=\left[\frac{1}{\tau}+\frac{t^{(2-\alpha)}}{\tau(2-\alpha)}-\frac{t^{(1-\alpha)}}{(1-\alpha)}\right],\qquad b=\left[\frac{1}{\tau}+\frac{t^{(2-\alpha)}}{\tau(2-\alpha)}\right]\text{,then}$$

$$u''_{i+1,\,j}=a\,u_{i+1,\,j}-f_{i+1,\,j}-b\,u_{i,\,j} \tag{2.3.2.4}$$

where

$u''_{i+1,\,j}=u''(x_j,t_{i+1})$, $u_{i+1,\,j}=u(x_j,t_{i+1})$, $u_{i,\,j}=u(x_j,t_i)$, and $f_{i+1,\,j}=f(x_j,t_{i+1})$.

Second the fourth order accurate finite difference estimate for $u''(x)$ is used from (2.3.1.5) to give

$$\delta_x^2 u_{i+1,\,j}=u''_{i+1,\,j}+\left(\frac{h^2}{12}\delta_x^2\right)\left(u''_{i+1,\,j}\right)+O(h^4). \tag{2.3.2.5}$$

Then, a compact (implicit) approximation for $u''(x)$ with fourth-order accuracy will be given as

$$u''_{i+1,\,j}=\frac{\delta_x^2 u_{i+1,\,j}}{\left(1+\frac{h^2}{12}\delta_x^2\right)}+O(h^4). \tag{2.3.2.6}$$

Using this estimate and considering the discrete solution of equation (2.3.2.4) which satisfies the approximation, we get

$$\left(\frac{12-a\,h^2}{12\,h^2}\right)\left[u_{i+1,\,j+1}+u_{i+1,\,j-1}\right]+\left[\frac{-a\,h^2-12+6\,a\,h^2}{6\,h^2}\right]u_{i+1,\,j}=\frac{-1}{12}\left[f_{i+1,\,j+1}+f_{i+1,\,j-1}\right]-$$
$$-\frac{5}{6}f_{i+1,\,j}-\frac{b}{h^2}\left[u_{i,\,j+1}+u_{i,\,j-1}\right]-\frac{5\,b}{6}u_{i,\,j}\,. \tag{2.3.2.7}$$

Then, we use collocation method to obtain system of algebraic equations which solved to compute the unknown function. Let $U_i(x)$ be a function that approximates $u(x,t_i)$ for the time-level $t_i=i\,\tau$, and is a linear combination of n+1 shape functions which is expressed as:

$$U_i(x)=\sum_{m=0}^{n}c_{mi}\,\varphi_m(x)\,, \tag{2.3.2.8}$$

where $\{c_{mi}\}_{m=0}^{n}$ are the unknown real coefficients, to be evaluated, and the $\varphi_m(x)$ are any knowing basis functions then equation (2.3.2.7) rewrite as

$$\left(\frac{12-a\,h^2}{12\,h^2}\right)\left[U_{i+1,\,j+1}+U_{i+1,\,j-1}\right]+\left[\frac{-a\,h^2-12+6\,a\,h^2}{6\,h^2}\right]U_{i+1,\,j}=\frac{-1}{12}\left[f_{i+1,\,j+1}+f_{i+1,\,j-1}\right]-$$

$$-\frac{5}{6}f_{i+1,\,j}-\frac{b}{h^2}\left[u_{i,\,j+1}+u_{i,\,j-1}\right]-\frac{5\,b}{6}u_{i,\,j},\quad i=0,1,2,\ldots,\quad j=1,2,\ldots,n-1. \tag{2.3.2.9}$$

Replacing U_i by the approximate solution given by equation (2.3.2.8) yields the following linear system of $n-1$ equations

$$\left(\frac{12-a\,h^2}{12\,h^2}\right)\left[\sum_{m=0}^{n}c_{mi}\,\varphi_{m\,j+1}+\sum_{m=0}^{n}c_{mi}\,\varphi_{m\,j-1}\right]+\left[\frac{a\,h^2-12-6\,a\,h^2}{6\,h^2}\right]\sum_{m=0}^{n}c_{mi}\,\varphi_{m\,j}=$$

$$=\frac{-1}{12}\left[f_{i+1,\,j+1}+f_{i+1,\,j-1}\right]-\frac{5}{6}f_{i+1,\,j}-\frac{b}{h^2}\left[u_{i,\,j+1}+u_{i,\,j-1}\right]-\frac{5\,b}{6}u_{i,\,j} \tag{2.3.2.10}$$

equivalently

$$\sum_{m=0}^{n}c_{mi}\left[d_1\,\varphi_{m\,j+1}+d_2\,\varphi_{m\,j}+d_1\,\varphi_{m\,j-1}\right]=$$

$$=d_3\left[f_{i+1,\,j+1}+f_{i+1,\,j-1}\right]+d_4\,f_{i+1,\,j}+d_5\left[u_{i,\,j+1}+u_{i,\,j-1}\right]+d_6\,u_{i,\,j}$$

(2.3.2.11)

where

$$d_1=\left(\frac{12-a\,h^2}{12\,h^2}\right)\qquad d_2=\left[\frac{-a\,h^2-12+6\,a\,h^2}{6\,h^2}\right]$$

$$d_3=\frac{-1}{12}\qquad d_4=\frac{-5}{6}$$

$$d_5=\frac{-b}{h^2}\qquad d_6=\frac{-5\,b}{6} \tag{2.3.2.12}$$

The system (2.3.2.11) consists of $(n-1)$ equation in the $(n+1)$ unknowns $\{c_{mi}\}_{m=0}^{n}$. To get a solution of this system we need two additional conditions. These conditions are obtained from the boundary conditions (2.1.2)

$$u(a,t_i)=\sum_{m=0}^{n}c_{m1}\varphi_m(a)=g_1(t_i),\qquad i=0,\ldots n \tag{2.3.2.13}$$

$$u(b,t_i)=\sum_{m=0}^{n}c_{m1}\varphi_m(b)=g_2(t_i),\qquad i=0,\ldots n \tag{2.3.2.14}$$

The system (2.3.2.11), equations (2.3.2.13) and (2.3.2.14) consist of $(n+1)$ equations in $(n+1)$ unknowns; this system is of the form

$$AC=F.$$

(2.3.2.15)

Upon solving the system (2.3.2.15), the function $u_i(x)$ is approximated by the sum:

$$u_i(x_j)=\sum_{m=0}^{n}c_{mi}\,\phi_m(x_j),\ j=0,1,2,\ldots,n. \tag{2.3.2.16}$$

1. Numerical Experiment

In this section, we illustrate the procedure of solving equations (2.1.1) - (2.1.3), which determines the solution of second-order linear weakly-singular Volterra integro-differential equations, by the following examples.

Example 1:

$f(x,t)$ is given so that the theoretical solution of this problem is $u(x,t) = x^2\, t^{\alpha}$. with $g_1(t) = 0$ $g_2(t) = t^{\alpha}$, and $k(t-s) = (t-s)^{-\alpha}$.

Example 2:

$f(x,t)$ is given so that the theoretical solution of this problem is $u(x,t) = (x^2+1)\, t^{\alpha}$. with $g_1(t) = t^{\alpha}$ $g_2(t) = 2\, t^{\alpha}$, and $k(t-s) = (t-s)^{-\alpha}$.

Example 3:

$f(x,t)$ is given so that the theoretical solution of this problem is $u(x,t) = t\, e^{x}$. with $g_1(t) = t$ $g_2(t) = t\, e$, and $k(t-s) = (t-s)^{-\alpha}$.

Table 1. Comparison between errors for example (1) at different values of *t* and α

x	Error = Exact Solution – Approximate Solution			
	$t = 0.5,\ \alpha = 0.0001$		$t = 0.99,\ \alpha = 0.99$	
	$\tau = 0.00001$	$\tau = 0.001$	$\tau = 0.001$	$\tau = 0.01$
0	0	0	0	0
0.1	2.225921E-012	2.592349E-010	2.048393E-008	7.836406E-007
0.2	8.890201E-012	9.202759E-010	7.579116E-008	1.974225E-006
0.3	1.999799E-011	2.021653E-009	1.680804E-007	3.738129E-006
0.4	3.554923E-011	3.563578E-009	2.972824E-007	6.117390E-006
0.5	5.554582E-011	5.546054E-009	4.633994E-007	9.074575E-006
0.6	7.995948E-011	7.969080E-009	6.664292E-007	1.245283E-005
0.7	1.090826E-010	1.083265E-008	9.064390E-007	1.580844E-005
0.8	1.399774E-010	1.413611E-008	1.181334E-006	1.793725E-005
0.9	2.021050E-010	1.775938E-008	1.556349E-006	1.558932E-005
1.0	0	0	0	0

Table 2. Comparison between errors for example (2) at different values of *t* and α

x	Error = Exact Solution – Approximate Solution			
	$t = 0.2,\ \alpha = 0.9$		$t = 0.88,\ \alpha = 0.000$	
	$\tau = 0.008$	$\tau = 0.08$	$\tau = 0.001$	$\tau = 0.0003$
0	0	0	0	0
0.1	2.054346E-005	7.726289E-004	1.469400E-005	4.662220E-006
0.2	2.769627E-005	1.361490E-003	1.494817E-005	4.463612E-006
0.3	3.108444E-005	1.794813E-003	1.566809E-005	4.702448E-006
0.4	3.368075E-005	2.087028E-003	1.667295E-005	5.002568E-006
0.5	3.635822E-005	2.239629E-003	1.796495E-005	5.390698E-006
0.6	3.922739E-005	2.241258E-003	1.954406E-005	5.864671E-006
0.7	4.180561E-005	2.067031E-003	2.141033E-005	6.428415E-006

0.8	4.227779E-005	1.677058E-003	2.355939E-005	7.028519E-006
0.9	3.471607E-005	1.014067E-003	2.635256E-005	8.417137E-006
1.0	0	0	0	0

Table 3. Comparison between errors for example (3) at different values of *t* and α

x	Error = Exact Solution – Approximate Solution			
	$t=1,\ \alpha=0.008$		$t=0.6,\ \alpha=0.008$	
	$\tau=0.001$	$\tau=0.01$	$\tau=0.03$	$\tau=0.0003$
0	0	0	0	0
0.1	3.116725E-010	2.025832E-009	3.715719E-009	7.433532E-011
0.2	3.390599E-010	2.777027E-009	6.063877E-009	7.726120E-011
0.3	3.748142E-010	3.211664E-009	7.684957E-009	8.570855E-011
0.4	4.142329E-010	3.585804E-009	8.896511E-009	9.470069E-011
0.5	4.577960E-010	3.966887E-009	9.817073E-009	1.046629E-010
0.6	5.059440E-010	4.363681E-009	1.041000E-008	1.156670E-010
0.7	5.591589E-010	4.736420E-009	1.046469E-008	1.278846E-010
0.8	6.177343E-010	4.907646E-009	9.507786E-009	1.405323E-010
0.9	6.960064E-010	4.192716E-009	6.608457E-009	1.674956E-010
1.0	0	0	0	0

Tables (1-3) display, the error for different values of *t* and α. It is observed that all the results of the proposed approximation for the new approach are in good agreement with the exact ones and exhibit the expected convergence.

9. Conclusion

We have reduced the solution of a class of linear weakly-singular partial integro-differential equations to the solution of partial differential equations by removing the singularity using an appropriate Taylor's approximation, then the discretization in time by the 2-point Euler backward finite difference method. After that the fourth-order accurate compact finite difference scheme for partial differential problems was developed. The method reduces the underlying problem to linear system of algebraic equations, which can be solved successively to obtain a numerical solution at varied time-levels We have considered several distinct examples to illustrate our new approach and have verified our solutions.

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